COLORADO SCHOOL OF MINES
ELECTRICAL ENGINEERING DEPARTMENT

EENG 577

ADVANCED ELECTRICAL MACHINE DYNAMICS FOR SMART-GRID SYSTEMS

M1-P1 Complex Numbers & Phasors

Complex Numbers

- Two parts
  - Real Part
  - Imaginary Part
- Rectangular Form
  - $n = a + jb$
  - a is the real part and b is the imaginary part
  - $j = \sqrt{-1}$
- Polar and Exponential Forms
  - $n = C\angle\phi° = Ce^{j\phi}$
  - C is the magnitude of the complex number and ϕ is the angle or phase of the complex number

Transformations

- Rectangular to Polar

$$C = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

- Euler's Identity

$$e^{\pm j\theta} = cos\theta \pm jsin\theta$$

- Polar to Rectangular

$$a = Ccos\theta \quad b = Csin\theta$$

Useful Identities

- $\pm j^2 = \mp 1$
- $j = \frac{1}{-j}$
- $\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{c^2+d^2}$

Mathematical Operations

- To add or subtract complex numbers, express them in rectangular form

$$n_1 = a_1 + jb_1 \qquad n_2 = a_2 + jb_2$$

$$n_1 \pm n_2 = (a_1 \pm a_2) + j(b_1 \pm b_2)$$

- To multiply or divide complex numbers, express them in polar form

$$n_1 = C_1 \angle \theta_1° \qquad n_2 = C_2 \angle \theta_2°$$

$$n_1 \cdot n_2 = (C_1 \cdot C_2) \angle (\theta_1° + \theta_2°)$$

$$\frac{n_1}{n_2} = \frac{C_1}{C_2} \angle (\theta_1° - \theta_2°)$$

Complex Number Sample Problems

1. Convert to polar and exponential forms:

z1 = 6 + *j*8

$$6+j8 = \sqrt{6^2+8^2}\ \angle \tan^{-1} 8/6 = \underline{10.0 \angle 53.13^\circ} = \underline{10 e^{j53.13^\circ}}$$

2. Convert the following complex numbers into rectangular form:

z1 = 12 ∠−60

$$12 \angle -60^\circ = 12\cos(-60) + j\,12\sin(-60) = \underline{6 - j10.39}$$

Complex Number Sample Problems

3. If A = 2 + *j*5 and B = 4 − *j*6,

find: A*(A + B)

$$A^*(A+B) = (2-j5)((2+j5)+(4-j6))$$

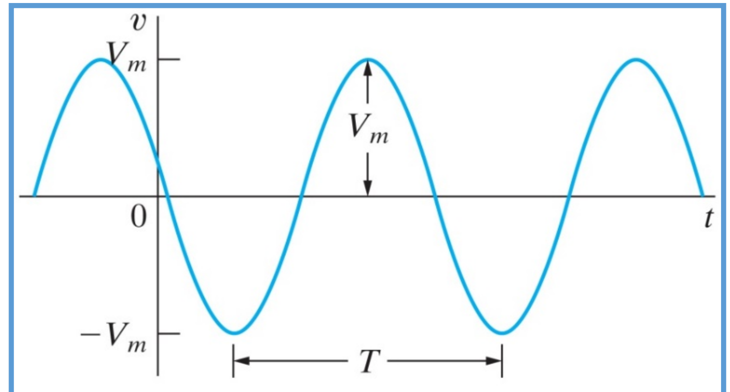
$$= 5.39\angle -68.20^\circ (6-j) = 5.39\angle -68.2^\circ (6.08\angle 9.46$$

$$= 32.76\angle -77.66^\circ = 7-j32$$

Sinusoid Steady-State Analysis

- Review the terminology of sinusoids

$$v(t) = V_m \cos(\omega t + \phi)\ \text{V}$$



Note

$$\sin(\omega t + \phi°) = \cos(\omega t + \phi° - 90°)$$

- V_m : maximum amplitude
- ω : angular frequency [rad/s]
- t : time [s]
- T: distance between 2 maxima or minima

- ϕ : phase angle [deg. or rad]
- ϕ positive (shift to the left)
- ϕ negative (shift to the right)
- $f = \omega / 2\pi$: frequency [Hz]
- $T = 1/f$: time period [s]

Root mean square (rms) value of a sinusoidal voltage source:

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

Sinusoid Steady-State Analysis

Phasor

- A complex number in polar form, with a magnitude and phase angle
- Derived from a sinusoid using the phasor transform

Euler's identity: $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

$\Rightarrow \quad \cos\phi = \mathrm{Re}\{e^{j\phi}\}$

Now consider

$$
\begin{aligned}
v(t) &= V_m\cos(\omega t + \phi) \\
&= V_m \mathrm{Re}\{e^{j(\omega t+\phi)}\} \\
&= V_m \mathrm{Re}\{e^{j\omega t}e^{j\phi}\} \\
&= \mathrm{Re}\{V_m e^{j\omega t}e^{j\phi}\}
\end{aligned}
$$

Two assumptions:

- *A cosine function*
- *A constant frequency f; where $\omega=2\pi f$*

Sinusoidal Steady-State Analysis & Phasors

Phasor transform

Extracts a sinusoid's magnitude and phase angle

Transforms a **function of time** into a **function of frequency**

$$\boldsymbol{V} = \mathcal{P}\{V_m \cos(\omega t + \phi)\} = V_m e^{j\phi} = V_m \angle \phi$$

Inverse phasor transform

- Turns a phasor back into a sinusoid, **if you know** the frequency

$$v(t) = \mathcal{P}^{-1}\{\mathbf{V}\} = \mathcal{P}^{-1}\{V_m \angle \phi\} = V_m \cos(\omega t + \phi)$$

Note the two assumptions:

- *A cosine function*
- *A constant frequency f; where $\omega = 2\pi f$*

Instantaneous power:

Given

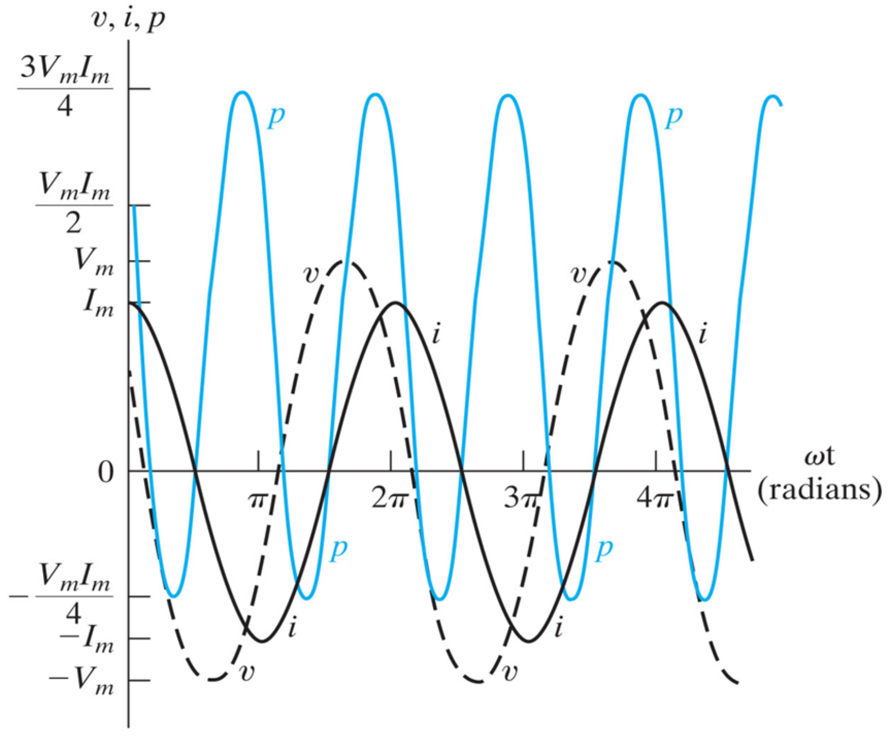
$$p(t) = v(t)\, i(t)$$
$$v(t) = V_m \cos(\omega t + \theta_V)$$
$$i(t) = I_m \cos(\omega t + \theta_I)$$

$$Note: \cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$p(t) = v(t) i(t)$$
$$= [V_m \cos(\omega t + \theta_v)][I_m \cos(\omega t + \theta_i)]$$

... *(lots of trigonometry)* ...

$$= \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t$$
$$- \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t$$



Rewrite

$$p(t) = P + P\cos 2\,\omega t - Q \sin 2\,\omega t$$

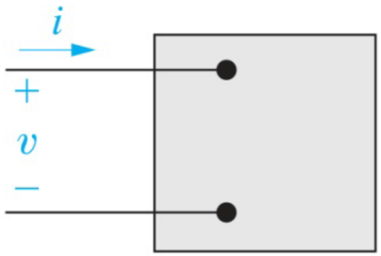
where.

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$ *The Real or Average power in W*

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$ *The Reactive power in VAR*

Example

Given:



$$i(t) = 20\cos(\omega t + 15°)\ \mathrm{A}$$
$$\mathrm{v(t)} = 100\cos(\omega t - 45°)\ \mathrm{V}$$

Find the average power, *P*, and the reactive power, *Q*

Expressing *i(t)* & *v(t)* in polar phasor form:

$\boldsymbol{I} = 20\angle 15°\ A \quad \& \quad \boldsymbol{V} = 100\angle -45°\ V$

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{(20)(100)}{2}\cos(-45° - 15°) = 500\ \mathrm{W}$$
$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) = \frac{(20)(100)}{2}\sin(-45° - 15°) = -866\ \mathrm{VAR}$$