COLORADO SCHOOL OF MINES
ELECTRICAL ENGINEERING DEPARTMENT

EENG 577

ADVANCED ELECTRICAL MACHINE DYNAMICS FOR SMART-GRID SYSTEMS

M2-P2 Transformer State Space Models

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Introduction to State-Space Equations Video - MATLAB

(mathworks.com)

Introduction to State-Space Equations Video - MATLAB

State Space Modeling

- The differential equations used to model the dynamic performance of an electromagnetic device in general are derived from the interaction between the device windings.
- For a device with "**n**" windings, one can write the voltage equation for each winding. A general expression of the following form can be used to express the terminal voltage, v_j of winding j:

$$\boldsymbol{v_j = r_j i_j + \frac{d}{dt}\{\lambda_j\}}$$

where, λ_j is the flux linkage of the j^{th} winding.

The flux linkage of the coil, λ_j, can be expressed in terms of the coil self-inductance, its current, as well as the mutual inductances and currents associated with all coupled windings, where r_j is the winding Ohmic resistance, i_j is the j_{th} winding instantaneous current.

- In a device with n windings, the flux linkage of the j^{th} winding can be expressed as follows:

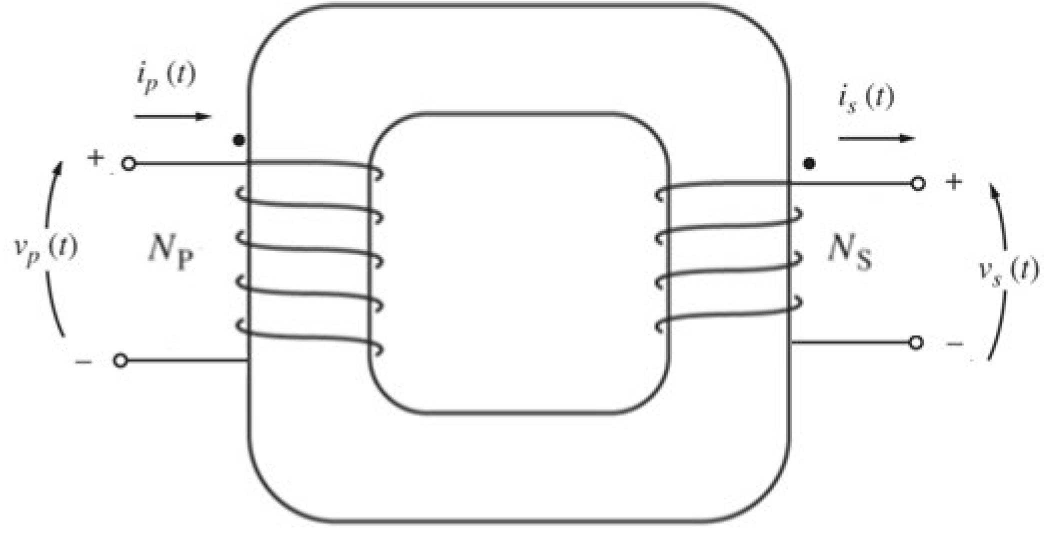
$$\boldsymbol{\lambda_j = \sum_{k=1}^{n} L_{jk}\, i_k}$$

- Using the above equations results in the following voltage expression for coil j:

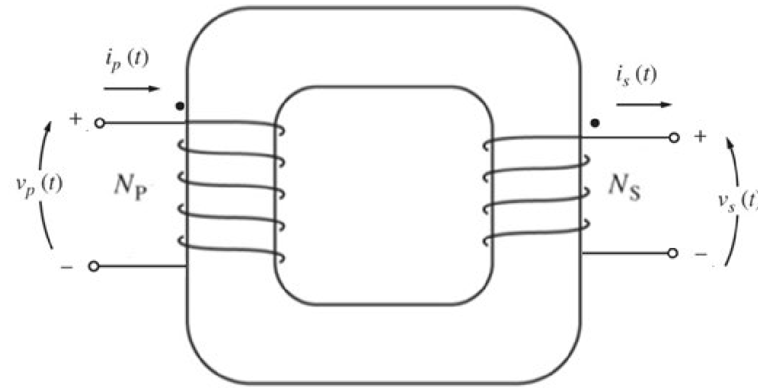
$$\boldsymbol{v_j = r_j i_j + L_{j1}\frac{di_1}{dt} + \ldots + L_{jn}\frac{di_n}{dt}}$$

I. Single Phase Transformer State Space Model

Consider a single-phase transformer with a primary winging "p" and secondary winding "s"



The voltage equations for a single-phase transformer with a primary winding "p" and secondary winding "s", and voltages v_p and v_s, can be expressed as follows:



$$v_p = r_p \cdot i_p + \frac{d\lambda_p}{dt}$$

$$v_s = r_s \cdot i_s + \frac{d\lambda_s}{dt}$$

Moreover, the flux linkage of the n and p windings can be expressed in terms of the winding's currents and self and mutual inductances as follows:

$$\lambda_p = L_{pp} \cdot i_p + L_{ps} \cdot i_s$$

$$\lambda_s = L_{ss} \cdot i_s + L_{sp} \cdot i_p$$

Based on the above, and assuming a linear unsaturated magnetic core, and assuming $\mathbf{L_{ps}} = \mathbf{L_{sp}} = \boldsymbol{M}$, the State Space, SS, model equations for the electromagnetic system considered can be represented as follows:

$$v_p = r_p\, i_p + L_{pp} \frac{d}{dt} i_p + M \frac{d}{dt} i_s$$

$$v_s = r_s\, i_s + M \frac{d}{dt} i_p + L_{ss} \frac{d}{dt} i_s$$

In matrix form:

$$\begin{bmatrix} v_p \\ v_s \end{bmatrix} = \begin{bmatrix} r_p & 0 \\ 0 & r_s \end{bmatrix} \begin{bmatrix} i_p \\ i_s \end{bmatrix} + \begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & L_{ss} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_p \\ i_s \end{bmatrix}$$

Now if we define:

A resistance matrix as **R**= $\begin{bmatrix} r_p & 0 \\ 0 & r_s \end{bmatrix}$

An inductance matrix as **L**= $\begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & L_{ss} \end{bmatrix}$

A current vector as **I**= $\begin{bmatrix} i_p \\ i_s \end{bmatrix}$

And a voltage vector as **V**= $\begin{bmatrix} v_p \\ v_s \end{bmatrix}$

The system SS model can be written as:

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{I} + \frac{d(\boldsymbol{L} \cdot \boldsymbol{I})}{dt}$$

- Assuming a linear unsaturated magnetic core,

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{I} + \boldsymbol{L}\frac{d(\boldsymbol{I})}{dt}$$

- Rearranging the equations and defining $\dot{\boldsymbol{I}} = \frac{d(\boldsymbol{I})}{dt}$ and $\boldsymbol{L}^{-1}$ as $1/\boldsymbol{L}$, one gets:

$$\dot{\boldsymbol{I}} = -\boldsymbol{L}^{-1} \cdot (\boldsymbol{R}) \cdot \boldsymbol{I} + \boldsymbol{L}^{-1} \cdot \boldsymbol{V}$$

- Solving this set of deferential equations would result in the values of the state

  space variables vector $\mathbf{I}=\begin{bmatrix} i_p \\ i_s \end{bmatrix}$

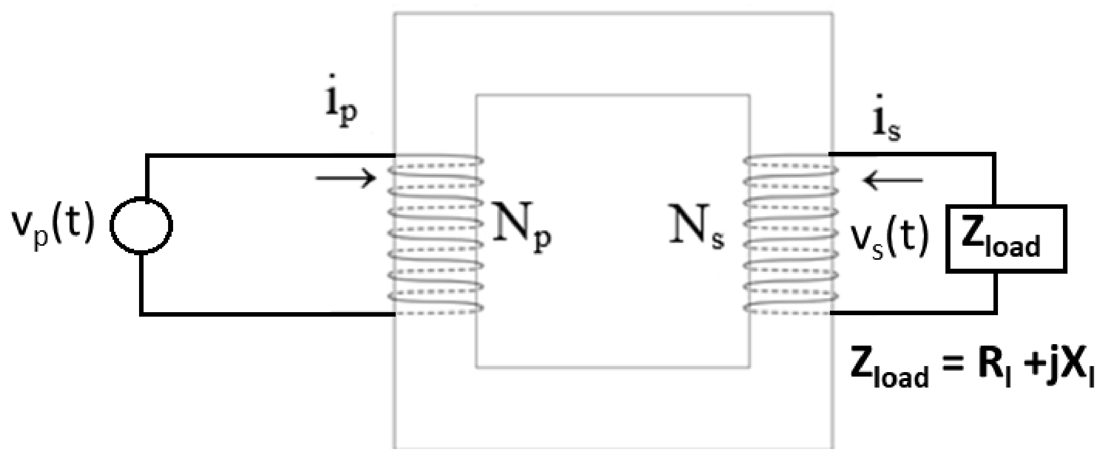
- The equation above can be written in a general form as follows:

$$\dot{\mathbf{X}} = \mathbf{A} \cdot \mathbf{X} + \mathbf{B} \cdot \mathbf{U}$$

- It can be solved using Euler's or Runge Kutta's Methods as shown below.

A Case Study

- Write the governing voltage equations for $v_p(t)$ and $v_s(t)$ in terms of currents $i_p(t)$ and $i_s(t)$ given $Z_{load}=R_l + jX_l$



- Assuming a linear unsaturated magnetic core,

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{I} + \boldsymbol{L}\frac{d(\boldsymbol{I})}{dt}$$

- Rearranging the equations and defining $\dot{\boldsymbol{I}} = \frac{d(\boldsymbol{I})}{dt}$ and $\boldsymbol{L^{-1}}$ as 1/***L***, one gets:

$$\dot{\boldsymbol{I}} = -\boldsymbol{L^{-1}} \cdot (\boldsymbol{R}) \cdot \boldsymbol{I} + \boldsymbol{L^{-1}} \cdot \boldsymbol{V}$$

$$\frac{d}{dt}\begin{bmatrix} i_p \\ i_s \end{bmatrix} = -\begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & L_{ss} \end{bmatrix}^{-1} \begin{bmatrix} r_p & 0 \\ 0 & r_s \end{bmatrix} \begin{bmatrix} i_p \\ i_s \end{bmatrix} + \begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & L_{ss} \end{bmatrix}^{-1} \begin{bmatrix} v_p \\ v_s \end{bmatrix}$$

- For the case when the secondary load, Z_{Load}, is of the inductive type (r_l & L_l), one can substitute $v_s = i_s\, Z_{Load} = r_l i_s + L_l \frac{di_s}{dt}$ into the above state space equation.

- After few steps, the state space model can be written as,

$$\frac{d}{dt}\begin{bmatrix} i_p \\ i_s \end{bmatrix} = -\begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & (L_{ss} + L_l) \end{bmatrix}^{-1} \begin{bmatrix} r_p & 0 \\ 0 & (r_s + r_l) \end{bmatrix} \begin{bmatrix} i_p \\ i_s \end{bmatrix} + \begin{bmatrix} L_{pp} & L_{ps} \\ L_{sp} & (L_{ss} + L_l) \end{bmatrix}^{-1} \begin{bmatrix} v_p \\ 0 \end{bmatrix}$$

- Solving this set of deferential equations would result in the values of the state space variables vector **I**= $\begin{bmatrix} i_p \\ i_s \end{bmatrix}$

Solution of State Space Equations

Euler's Method

$$\dot{y} = \frac{dy}{dt} = f(t, y), y(0) = y0$$

Input t_0 and y_0.

Input step size, h and the number of steps, n.

for jj from 1 to n **do**

a. $t_{n+1}=t_n+h$

b. $y_{n+1}= y_n + h \times f(t_n,y_n)$

c. Print t and y

end

State Space Model Sample Matlab Code Using Euler's Method:

```
%% State Space Model (Euler's Method)
R = [0.0423  0;0 (0.1172+40.922)];
L = [0.0276  0.062; 0.062  (0.14467+0.0814)];
Vp = sqrt(2)*120;

n =60;
h=0.000167;                          % step size
t = 0:h:0.1;                    % the range of t
x = zeros(2,length(t));         % Calculates upto x(360)

K = [1;0];
A = -inv(L)*R;
B = inv(L);
u = K*Vp;
dx =@(t,x)(A*x+ B*u*sin(377*t+pi/2));
for i = 1:(length(t)-1)
    t(i+1) = t(i)+h;
    x(:,i+1) = x(:,i) + h * dx(t(i),x(:,i));
end
plot(t,x(1,:),t,x(2,:))
legend({'I1','I2'})
xlabel('time')
ylabel('Current Values')
title('Euler Method')
```

Solution of State Space Equations

4th-Order Runge Kutta's Method for ODEs

$$\dot{\boldsymbol{y}} = \frac{dy}{dt} = f(t, y), y(0) = y0$$

Input t_0 and y_0.

input step size, h and the number of steps, n.

for jj from 1 to n **do**

a. $k1 = f(t_n, y_n)$

b. $k2 = f(t_n + \frac{h}{2},\ y_n + h \times \frac{k1}{2})$

c. $k3 = f(t_n + \frac{h}{2}, y_n + h \times \frac{k2}{2})$

d. $k4 = f(t_n + h, y_n + h \times k3)$

e. $y_{n+1} = y_n + \frac{1}{6} \times h \times (k_1 + k_2 + k_3 + k_4)$

f. Print t and y

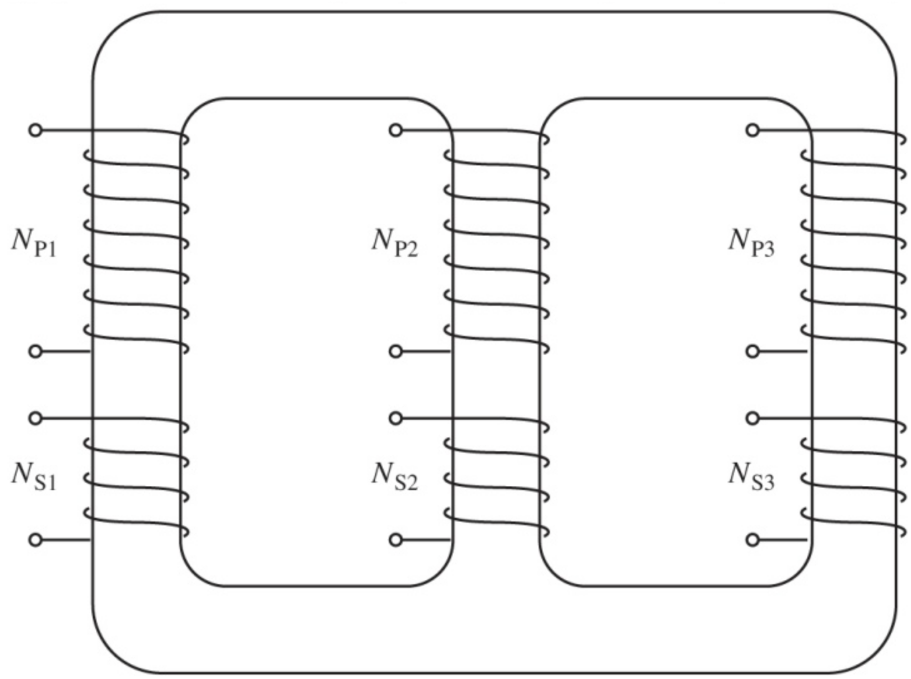
end

State Space Model (4th-Order Runge Kutta Method for ODEs)

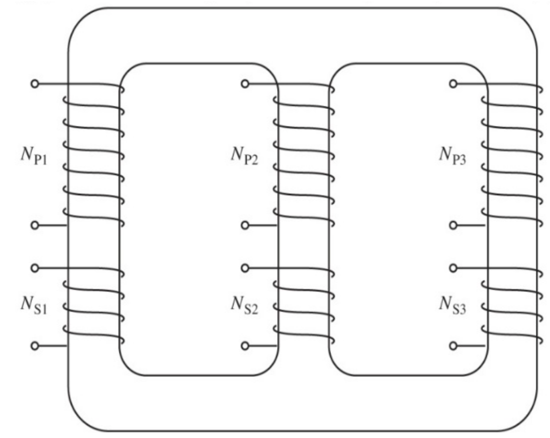
```
%% State Space Model (4th-Order Runge Kutta Method for ODEs)
tspan = [0 0.167]; % time step
iniCon = [0;0];    % Initial value of state

[t,y] = ode45(@myode, tspan, iniCon);
plot(t,y(:,1), t,y(:,2));
function dx = myode(t,x)
R = [0.042 0;0 (0.117+40.922)];
L = [0.0276 0.0629; 0.0629 (0.14467+0.0814)];
Vp = 120*sqrt(2);
A = -inv(L)*R;
B = inv(L);
K = [Vp;0];
u = K*sin(377*t+pi/2);
dx = A*x+ B*u;
end
```

II. Three Phase Transformers

Consider a three-phase transformer with six windings:



The Flux linkages for the coils are as follows:



$$\lambda_{p1} = L_{p1p1} * i_{p1} + L_{p1p2} * i_{p2} + L_{p1p3} * i_{p3} + L_{p1s1} * i_{s1} + L_{p1s2} * i_{s2} + L_{p1s3} * i_{s3}$$

$$\lambda_{p2} = L_{p2p1} * i_{p1} + L_{p2p2} * i_{p2} + L_{p2p3} * i_{p3} + L_{p2s1} * i_{s1} + L_{p2s2} * i_{s2} + L_{p2s3} * i_{s3}$$

$$\lambda_{p3} = L_{p3p1} * i_{p1} + L_{p3p2} * i_{p2} + L_{p3p3} * i_{p3} + L_{p3s1} * i_{s1} + L_{p3s2} * i_{s2} + L_{p3s3} * i_{s3}$$

$$\lambda_{s1} = L_{s1p1} * i_{p1} + L_{s1p2} * i_{p2} + L_{s1p3} * i_{p3} + L_{s1s1} * i_{s1} + L_{s1s2} * i_{s2} + L_{s1s3} * i_{s3}$$

$$\lambda_{s2} = L_{s2p1} * i_{p1} + L_{s2p2} * i_{p2} + L_{s2p3} * i_{p3} + L_{s2s1} * i_{s1} + L_{s2s2} * i_{s2} + L_{s2s3} * i_{s3}$$

$$\lambda_{s3} = L_{s3p1} * i_{p1} + L_{s3p2} * i_{p2} + L_{s3p3} * i_{p3} + L_{s3s1} * i_{s1} + L_{s3s2} * i_{s2} + L_{s3s3} * i_{s3}$$

In this case, the subscripts 1, 2, and 3 correspond to phases a, b, and c. As such the self and mutual inductances are as follows:

- L_{p1p1}, L_{p2p2}, L_{p3p3} are the primary self inductances for the a, b, and c phase coils respectively
- L_{s1s1}, L_{s2s2}, L_{s3s3} are the secondary self inductances for the a, b, and c phase coils respectively
- L_{p1s1}, L_{p1s2}, L_{p1s3} are the mutual inductances primary a phase winding and secondary a, b, and c phase windings respectively
- L_{s1p1}, L_{s1p2}, L_{s1p3} are the mutual inductances between the secondary a phase and primary a, b, and c phase windings respectively
- L_{p2s1}, L_{p2s2}, L_{p2s3} are the mutual inductances between the primary b phase winding and secondary a, b, and c phase windings

- L_{s2p1}, L_{s2p2}, L_{s2p3} are the mutual inductances between the secondary b phase and primary a, b, and c phase windings respectively
- L_{p3s1}, L_{p3s2}, L_{p3s3} are the mutual inductances between the primary c phase winding and secondary a, b, and c phase windings
- L_{s3p1}, L_{s3p2}, L_{s3p3} are the mutual inductances between the secondary c phase and primary a, b, and c phase windings respectively
- L_{p1p2}, L_{p1p3} are the mutual inductances between the primary a phase winding and the primary b and c phase windings respectively
- L_{p2p1}, L_{p2p3} are the mutual inductances between the primary b phase winding and the primary a and c phase windings respectively
- L_{p3p1}, L_{p3p2} are the mutual inductances between the primary c phase winding and the primary a and b phase windings respectively

- L_{s1s2}, L_{s1s3} are the mutual inductances between the secondary a phase winding and the secondary b and c phase windings respectively
- L_{s2s1}, L_{s2s3} are the mutual inductances between the secondary b phase winding and the secondary a and c phase windings respectively
- L_{s3s1}, L_{s3s2} are the mutual inductances between the secondary c phase winding and the secondary a and b phase windings respectively

Also, the winding currents are as follows:

- i_{p1}, i_{p2}, i_{p3} are the currents in the primary a, b, and c phase windings respectively.
- i_{s1}, i_{s2}, i_{s3} are the currents in the secondary a, b, and c phase windings respectively.

Three-Phase Transformer State-Space (SS) Model

- The single-phase transformer state space model presented earlier is extended and applied to the case of a three-phase transformer.
- Again, the SS Model governing the transformer performance can be expressed as:

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{I} + \frac{d(\boldsymbol{L} \cdot \boldsymbol{I})}{dt}$$

Where for the 3-phase transformer, the primary windings are p1, p2, and p3. Also, the secondary windings are s1, s2, and s2.
As such, the following can be defined:

V = $\begin{bmatrix} v_{p1} \\ v_{p2} \\ v_{p3} \\ v_{s1} \\ v_{s2} \\ v_{s3} \end{bmatrix}$ is the voltage Vector,

I = $\begin{bmatrix} i_{p1} \\ i_{p2} \\ i_{p3} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{bmatrix}$ is the current Vector,

$$\mathbf{R} = \begin{bmatrix} r_{p1} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{p3} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{s1} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{s2} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{s3} \end{bmatrix}$$

is the resistance matrix,

and

$$\mathbf{L} = \begin{bmatrix} L_{p1p1} & L_{p1p2} & L_{p1p3} & L_{p1s1} & L_{p1s2} & L_{p1s3} \\ L_{p2p1} & L_{p2p2} & L_{p2p3} & L_{p2s1} & L_{p2s2} & L_{p2s2} \\ L_{p3p1} & L_{p3p2} & L_{p3p3} & L_{p3s1} & L_{p3s2} & L_{p3s3} \\ L_{s1p1} & L_{s1p2} & L_{s1p3} & L_{s1s1} & L_{s1s2} & L_{s1s3} \\ L_{s2p1} & L_{s2p2} & L_{s2p3} & L_{s2s1} & L_{s2s2} & L_{s2s3} \\ L_{s3p1} & L_{s3p2} & L_{s3p3} & L_{s3s1} & L_{s3s2} & L_{s3s3} \end{bmatrix}$$

is the inductance matrix

This results in the following state space equation:

$$\begin{pmatrix} v_{p1} \\ v_{p2} \\ v_{p3} \\ v_{s1} \\ v_{s2} \\ v_{s3} \end{pmatrix} = \begin{pmatrix} r_{p1} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{p3} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{s1} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{s2} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{s3} \end{pmatrix} \begin{pmatrix} i_{p1} \\ i_{p2} \\ i_{p3} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{pmatrix} +$$

$$\frac{d}{dt} \left\{ \begin{pmatrix} L_{p1p1} & L_{p1p2} & L_{p1p3} & L_{p1s1} & L_{p1s2} & L_{p1s3} \\ L_{p2p1} & L_{p2p2} & L_{p2p3} & L_{p2s1} & L_{p2s2} & L_{p2s2} \\ L_{p3p1} & L_{p3p2} & L_{p3p3} & L_{p3s1} & L_{p3s2} & L_{p3s3} \\ L_{s1p1} & L_{s1p2} & L_{s1p3} & L_{s1s1} & L_{s1s2} & L_{s1s3} \\ L_{s2p1} & L_{s2p2} & L_{s2p3} & L_{s2s1} & L_{s2s2} & L_{s2s3} \\ L_{s3p1} & L_{s3p2} & L_{s3p3} & L_{s3s1} & L_{s3s2} & L_{s3s3} \end{pmatrix} \begin{pmatrix} i_{p1} \\ i_{p2} \\ i_{p3} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{pmatrix} \right\}$$

Assuming a linear unsaturated magnetic core,

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{I} + \boldsymbol{L} \frac{\boldsymbol{d(I)}}{\boldsymbol{dt}}$$

Rearranging the equations and defining $\dot{I} = \frac{d(I)}{dt}$ and L^{-1} as 1/*L*, one gets:

$$\dot{\boldsymbol{I}} = -\boldsymbol{L}^{-\boldsymbol{1}} \cdot \boldsymbol{R} \cdot \boldsymbol{I} + \boldsymbol{L}^{-\boldsymbol{1}} \cdot \boldsymbol{V}$$

It can be appreciated that the main parameters of the above SS model are the winding inductances. These inductances can be determined from measurements or from computational electromagnetic Finite Element, FE, field solutions.

Solving this set of deferential equations would result in the values of the state space variables given in vector ***I***.

The equation above can be written in a general form as follows:

$$\dot{\mathrm{X}} = \mathrm{A} \cdot \mathrm{X} + \mathrm{B} \cdot \mathrm{U}$$

It can be solved using MATLAB and the Euler’s or Runga Kutta’s Methods.

Three-Phase Transformer with an R-L Load SS Model

The above SS model can be used when the transformer is connected to a load, where the secondary windings' voltages are related to a load, Z_{load}, of components r_l and L_l . In such case, the secondary voltages are expressed in terms of the load as:

$$v_{s1} = -(r_l i_{s1} + L_l \frac{di_{s1}}{dt})$$

$$v_{s2} = -(r_l i_{s2} + L_l \frac{di_{s2}}{dt})$$

$$v_{s3} = -(r_l i_{s3} + L_l \frac{di_{s3}}{dt})$$

$$\frac{d}{dt}\begin{pmatrix} i_{p1} \\ i_{p2} \\ i_{p3} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{pmatrix}$$

$$= -\begin{pmatrix} L_{p1p1} & L_{p1p2} & L_{p1p3} & L_{p1s1} & L_{p1s2} & L_{p1s3} \\ L_{p2p1} & L_{p2p2} & L_{p2p3} & L_{p2s1} & L_{p2s2} & L_{p2s2} \\ L_{p3p1} & L_{p3p2} & L_{p3p3} & L_{p3s1} & L_{p3s2} & L_{p3s3} \\ L_{s1p1} & L_{s1p2} & L_{s1p3} & L_{s1s1}+L_l & L_{s1s2} & L_{s1s3} \\ L_{s2p1} & L_{s2p2} & L_{s2p3} & L_{s2s1} & L_{s2s2}+L_l & L_{s2s3} \\ L_{s3p1} & L_{s3p2} & L_{s3p3} & L_{s3s1} & L_{s3s2} & L_{s3s3}+L_l \end{pmatrix}^{-1}$$

$$* \begin{pmatrix} r_{p1} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{p3} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{s1}+r_l & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{s2}+r_l & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{s2}+r_l \end{pmatrix} * \begin{pmatrix} i_{p1} \\ i_{p2} \\ i_{p3} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{pmatrix}$$

$$+ \begin{pmatrix} L_{p1p1} & L_{p1p2} & L_{p1p3} & L_{p1s1} & L_{p1s2} & L_{p1s3} \\ L_{p2p1} & L_{p2p2} & L_{p2p3} & L_{p2s1} & L_{p2s2} & L_{p2s2} \\ L_{p3p1} & L_{p3p2} & L_{p3p3} & L_{p3s1} & L_{p3s2} & L_{p3s3} \\ L_{s1p1} & L_{s1p2} & L_{s1p3} & L_{s1s1}+L_l & L_{s1s2} & L_{s1s3} \\ L_{s2p1} & L_{s2p2} & L_{s2p3} & L_{s2s1} & L_{s2s2}+L_l & L_{s2s3} \\ L_{s3p1} & L_{s3p2} & L_{s3p3} & L_{s3s1} & L_{s3s2} & L_{s3s3}+L_l \end{pmatrix}^{-1} \begin{pmatrix} v_{p1} \\ v_{p2} \\ v_{p3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this set of deferential equations using the using Euler's or Runge Kutta's Methods in MATLAB, as was showed earlier, would result in the values of the state space variables vector ***I***.