COLORADO SCHOOL OF MINES
ELECTRICAL ENGINEERING DEPARTMENT

EENG 577

ADVANCED ELECTRICAL MACHINE DYNAMICS FOR SMART-GRID SYSTEMS

M6-2 Induction Machine State Space Models

Dr. A.A. Arkadan

An Overview

In this module, induction machine state space models are developed for the following:

1. The time domain, abc, frame of reference.
2. The conventional DQ0 frame of reference

<u>Note:</u> This presentation is based on section 3 of a set of class notes developed for induction machines. As such, all equations and figures have a prefix “3” as shown in the text.

NETWORK MODEL FOR INDUCTION MOTORS IN D-Q FRAME OF REFERENCE

Consider the idealized three phase, two pole induction motor shown in Figure (3-1-a). Coils A, B, and C represent coils of stator phases A, B, and C respectively. The coils a, b, and c represent an equivalent three rotor phases a, b, and c respectively. Only one equivalent turn is shown which serves to locate the axis of each winding. The three winding axis of the three stator (and three rotor) windings are mutually displaced by 120°. The stator A-Axis is displaced from the rotor a-axis by the angle σ. The D-Q symmetrical model of a polyphase machine has the following properties:

1. Uniform air gap.
2. The stator windings are identical and arranged in a way such as balanced stator currents produce one sinusoidal MMF wave in space with the phases.
3. The rotor bars or coils are arranged or distributed in a way such that the rotor MMF is a space sinusoid having the same number of poles as that of the stator MMF wave.
4. Iron saturation is neglected (linear magnetic circuit).

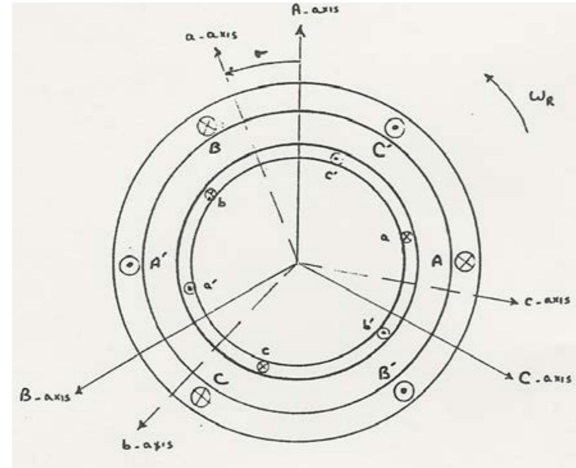


Fig. 3-1 Induction Machine
(a) Schematic Diagram

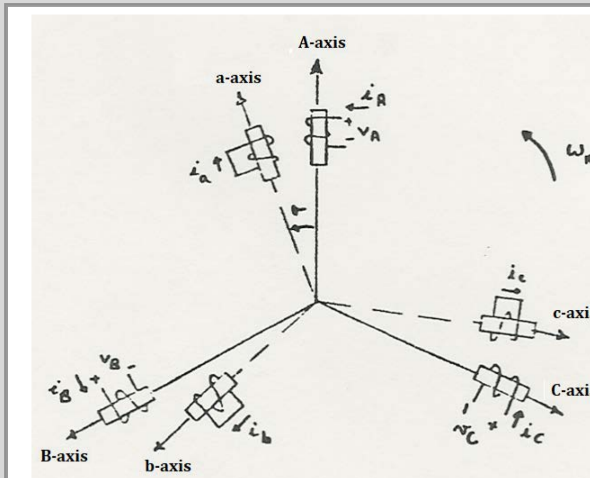


Fig. 3-1 Induction Machine
(b) Equivalent Winding Representation

Although the model is different from an actual machine, it still offers a good tool in studying the performance of many induction machines, and is being adopted here.

Next the differential equations describing the motor are written. From Figure (3.1) for coil A the terminal voltage v_A can be expressed as follows:

$$v_A = r_A i_A + \frac{d}{dt} \lambda_A \qquad (3.1)$$

or can be expanded in terms of the different coils inductances and currents as follows:

$$v_A = r_A i_A + \frac{d}{dt} (L_{AA} i_A + L_{AB} i_B + L_{AC} i_C + L_{Aa} i_a + L_{Ab} i_b + L_{Ac} i_c \qquad (3.2)$$

where r_A is phase A resistance in ohms [Ω], L_{AA} is phase A self inductance in Henrys [H], L_{AB}, L_{AC}, L_{Aa}, L_{Ab}, and L_{Ac} represent the mutual inductances between coil A and the other coils in Henrys. Doing so for all the coils, one obtains a set of differential equations that describe the behavior of the currents in the different coils. If the matrix notation representation is used, in which a bar (_) under the letter represents a vector or a matrix, one can write the following:

$$
\begin{bmatrix} v_A \\ v_B \\ v_C \\ v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} r_A & & & & & \underline{O} \\ & r_B & & & & \\ & & r_C & & & \\ & & & r_a & & \\ & & & & r_b & \\ & \underline{O} & & & & r_c \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \\ i_a \\ i_b \\ i_c \end{bmatrix} +
$$

$$
\frac{d}{dt} \left\{ \begin{bmatrix} L_{AA} & L_{AB} & L_{AC} & L_{Aa} & L_{Ab} & L_{Ac} \\ L_{BA} & L_{BB} & L_{BC} & L_{Ba} & L_{Bb} & L_{Bc} \\ L_{CA} & L_{CB} & L_{CC} & L_{Ca} & L_{Cb} & L_{Cc} \\ L_{aA} & L_{aB} & L_{aC} & L_{aa} & L_{ab} & L_{ac} \\ L_{bA} & L_{bB} & L_{bC} & L_{ba} & L_{bb} & L_{bc} \\ L_{cA} & L_{cB} & L_{cC} & L_{ca} & L_{cb} & L_{cc} \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \\ i_a \\ i_b \\ i_c \end{bmatrix} \right\} \quad (3.3)
$$

In a compact matrix notation one can write:

$$\begin{bmatrix} \underline{V}_{ABC} \\ \underline{V}_{abc} \end{bmatrix} = \begin{bmatrix} \underline{R}_{ss} & \underline{0} \\ \underline{0} & \underline{R}_{rr} \end{bmatrix} \begin{bmatrix} \underline{I}_{ABC} \\ \underline{I}_{abc} \end{bmatrix} + \frac{d}{dt} \left\{ \begin{bmatrix} \underline{L}_{ss} & \underline{L}_{sr} \\ \underline{L}_{sr}^{t} & \underline{L}_{rr} \end{bmatrix} \begin{bmatrix} \underline{I}_{ABC} \\ \underline{I}_{abc} \end{bmatrix} \right\} \quad (3.4)$$

From Figure (3.1.a) the inductance coefficients L_{AA}, L_{BB}, and L_{CC} are equal since these windings have identical flux paths (neglecting slot effects), then one can write:

$$L_{AA} = L_{BB} = L_{CC} = L_{ss} \quad (3.5)$$

Also since the machine has a cylindrical rotor, then:

$$L_{AB} = L_{BC} = L_{AC} = L_{sm} \quad (3.6)$$

For rotor quantities $L_{aa} = L_{bb} = L_{cc} = L_{rr}$ since they are independent of rotor position.

Similarly, $L_{ab} = L_{bc} = L_{ac} = L_{rm}$ (3.7)

Then one can write:

$$\underline{L}_{ss} = \begin{bmatrix} L_{ss} & L_{sm} & L_{sm} \\ L_{sm} & L_{ss} & L_{sm} \\ L_{sm} & L_{sm} & L_{ss} \end{bmatrix} \tag{3.8}$$

$$\underline{L}_{rr} = \begin{bmatrix} L_{rr} & L_{rm} & L_{rm} \\ L_{rm} & L_{rr} & L_{rm} \\ L_{rm} & L_{rm} & L_{rr} \end{bmatrix} \tag{3.9}$$

where the entries of $\underline{L}_{ss}$ and $\underline{L}_{rr}$ are assumed to be constant.

The coefficients of the inductances forming the matrix $\underline{L}_{sr}$, in contrary to the previous cases, are dependent on the rotor position. It can be deduced from Figure (3.1) that these coefficients vary cosinusoidally with the rotor angular position (σ) with some phase differences. The following can be written for these coefficients:

$$L_{Aa} = L_{Bb} = L_{Cc} = L_{sr} \cos(\sigma)$$
$$L_{Ab} = L_{Bc} = L_{Ca} = L_{sr} \cos(\sigma + 2\pi/3) \qquad (3.10)$$
$$L_{Ac} = L_{Ba} = L_{Cb} = L_{srm} \cos(\sigma + 4\pi/3)$$

where $\sigma = \sigma_0 + \omega_R t$.

and σ_0 is the angle at time t=0 between the stator A-axis and the rotor a-axis in radians.

Hence $\underline{L}_{sr}$ can be written as follows:

$$\underline{L}_{sr} = L_{sr} \begin{bmatrix} \cos(\sigma) & \cos(\sigma + 2\pi/3) & \cos(\sigma + 4\pi/3) \\ \cos(\sigma + 4\pi/3) & \cos(\sigma) & \cos(\sigma + 2\pi/3) \\ \cos(\sigma + 2\pi/3) & \cos(\sigma + 4\pi/3) & \cos(\sigma) \end{bmatrix} \qquad (3.11)$$

Since we are dealing with 3-phase balanced machines, one can write the following for the rotor and stator phase resistances:

$r_s = r_A = r_B = r_C$ and $r_r = r_a = r_b = r_c$

From equation (3.3) the resistance submatrices $\underline{R}_{ss}$ and $\underline{R}_{rr}$ can be written as follows:

$$\underline{R}_{ss} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \tag{3.12}$$

$$\underline{R}_{rr} = \begin{bmatrix} r_r & 0 & 0 \\ 0 & r_r & 0 \\ 0 & 0 & r_r \end{bmatrix} \tag{3.13}$$

Using the fact that equations (3.8) and (3.9) have constant coefficients equation (3.4) can be rewritten as:

$$\begin{bmatrix} \underline{V}_{ABC} \\ \underline{V}_{abc} \end{bmatrix} = \begin{bmatrix} \underline{R}_{ss} & \underline{0} \\ \underline{0} & \underline{R}_{rr} \end{bmatrix} \begin{bmatrix} \underline{I}_{ABC} \\ \underline{I}_{abc} \end{bmatrix} + \underbrace{\begin{bmatrix} \underline{0} & \dot{\underline{L}}_{sr} \\ \dot{\underline{L}}^{t}_{sr} & \underline{0} \end{bmatrix} \cdot \begin{bmatrix} \underline{I}_{ABC} \\ \underline{I}_{abc} \end{bmatrix}}_{\text{Rotational Voltage}} + \underbrace{\begin{bmatrix} \underline{L}_{ss} & \underline{L}_{sr} \\ \underline{L}^{t}_{sr} & \underline{L}_{rr} \end{bmatrix} \cdot \begin{bmatrix} \dot{\underline{I}}_{ABC} \\ \dot{\underline{I}}_{abc} \end{bmatrix}}_{\text{Transformer Voltage}} \quad (3.14)$$

where the dot (·) represents a derivative with respect to time.

In terms of flux linkage equation (3.14) can be written as:

$$\begin{bmatrix} \underline{V}_{ABC} \\ \underline{V}_{abc} \end{bmatrix} = \begin{bmatrix} \underline{R}_{ss} & 0 \\ 0 & \underline{R}_{rr} \end{bmatrix} \begin{bmatrix} \underline{I}_{ABC} \\ \underline{I}_{abc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \underline{\Lambda}_{ABC} \\ \underline{\Lambda}_{abc} \end{bmatrix} \quad (3.15)$$

Since most network software packages can handle constant inductances only, equation (3.14) can not be used to derive an equivalent network model for the induction machine. A transformation is needed to get a model with constant inductances. It has been proven useful to transform the equation (3.14) which describes the behavior of the machine to D-Q frame of reference. However, since the rotor mmf and stator mmf in an induction machine rotate at different speeds because of the slip, different transformation matrices are required, one for each.

The Park's Transformation will be used, and the D-Q axis can be fixed with respect to one of the following frame of references:

a) The stationary reference frame (at stator).

b) The synchronously rotating reference frame.

c) The reference frame fixed at the rotor.

So as shown in equations (3.11) and (3.14), where actual voltages and currents are being used, the mutual inductances are time varying (dependent or the rotor position). However, when the currents and voltages are being transformed to D-Q axes, the resulting differential equations will have a set of constant coefficients when the rotor speed is constant.

In this set of notes, in order to transform the time varying inductances to linear inductances (time independent inductances), the D-Q frame of reference was chosen to be fixed at the rotor, however, the other two references can be chosen as well. Consequently the following transformation matrices will be used for the stator and rotor respectively:

$$\underline{T}_s = \frac{2}{3}\begin{bmatrix} \cos(\sigma) & \cos(\sigma - 2\pi/3) & \cos(\sigma - 4\pi/3) \\ -\sin(\sigma') & -\sin(\sigma - 2\pi/3) & -\sin(\sigma - 4\pi/3) \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \quad (3.16)$$

and

$$\underline{T}_r = 2/3 \begin{bmatrix} \cos(0) & \cos(-2\pi/3) & \cos(-4\pi/3) \\ -\sin(0) & -\sin(-2\pi/3) & -\sin(-4\pi/3) \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \quad (3.17)$$

where $\sigma = \sigma_o + \omega_r t$ is shown in Figure (3.1) and σ_o is the angle between the stator A-axis and rotor a-axis was chosen equal to zero at t = 0 sec, and ω_r is the speed of the rotor.

The inverses of $\underline{T}_s$ and $\underline{T}_r$ were found to be as follows:

$$\underline{T}_s^{-1} = \begin{bmatrix} \cos\sigma & -\sin\sigma & 1 \\ \cos(\sigma-2\pi/3) & -\sin(\sigma-2\pi/3) & 1 \\ \cos(\sigma-4\pi/3) & -\sin(\sigma-4\pi/3) & 1 \end{bmatrix} \qquad (3.18)$$

and

$$\underline{T}_r^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1/2 & \sqrt{3}/2 & 1 \\ -1/2 & -\sqrt{3}/2 & 1 \end{bmatrix} \qquad (3.19)$$

$\underline{T}_s$ and $\underline{T}_r$ were chosen such that:

$$\begin{aligned} \underline{I}_{DQO} &= \underline{T}_s \, \underline{I}_{ABC} \\ \underline{V}_{DQO} &= \underline{T}_s \, \underline{V}_{ABC} \\ \underline{\Lambda}_{DQO} &= \underline{T}_s \, \underline{\Lambda}_{ABC} \end{aligned} \qquad (3.20)$$

and

$$
\begin{aligned}
\underline{I}_{dqo} &= \underline{T}_r \, \underline{I}_{abc} \\
\underline{V}_{dqo} &= \underline{T}_r \, \underline{V}_{abc} \\
\underline{\Lambda}_{dqo} &= \underline{T}_r \, \underline{\Lambda}_{abc}
\end{aligned}
\tag{3.21}
$$

Using equation (3.20) and (3.21) in equation (3.15) and after some matrix algebra and substituting for the flux linkages in terms of the inductance matrices and current vectors, the transformed induction motor model becomes:

$$
\begin{aligned}
\begin{bmatrix} \underline{V}_{DQO} \\ \underline{V}_{dqo} \end{bmatrix} =&
\begin{bmatrix} (\underline{T}_s \cdot \underline{R}_{ss} \cdot \underline{T}_s^{-1}) & \underline{0} \\ \underline{0} & (\underline{T}_r \cdot \underline{R}_{rr} \cdot \underline{T}_r^{-1}) \end{bmatrix} \cdot
\begin{bmatrix} \underline{I}_{DQO} \\ \underline{I}_{dqo} \end{bmatrix} + \\
& \frac{d}{dt} \left\{ \begin{bmatrix} (\underline{T}_s \cdot \underline{L}_{ss} \cdot \underline{T}_s^{-1}) & (\underline{T}_s \cdot \underline{L}_{sr} \cdot \underline{T}_r^{-1}) \\ (\underline{T}_r \cdot \underline{L}_{sr}^{t} \cdot \underline{T}_s^{-1}) & (\underline{T}_r \cdot \underline{L}_{rr} \cdot \underline{T}_r^{-1}) \end{bmatrix}
\begin{bmatrix} \underline{I}_{DQO} \\ \underline{I}_{dqo} \end{bmatrix} \right\} - \\
& \begin{bmatrix} (\dot{\underline{T}}_s \cdot \underline{L}_{ss} \cdot \underline{T}_s^{-1}) & (\dot{\underline{T}}_s \cdot \underline{L}_{sr} \cdot \underline{T}_r^{-1}) \\ (\dot{\underline{T}}_r \cdot \underline{L}_{sr}^{t} \cdot \underline{T}_s^{-1}) & (\dot{\underline{T}}_r \cdot \underline{L}_{rr} \cdot \underline{T}_r^{-1}) \end{bmatrix}
\begin{bmatrix} \underline{I}_{DQO} \\ \underline{I}_{dqo} \end{bmatrix}
\end{aligned}
\tag{3.22}
$$

Then one can find the following by using the expressions given above for the different matrices:

$$\dot{\underline{T}}_s \cdot \underline{L}_{ss} \cdot \underline{T}_s^{-1} = \begin{bmatrix} 0 & \omega_r(L_{ss} - L_{sm}) & 0 \\ -\omega_r(L_{ss} - L_{sm}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.30)$$

similarly,

$$\dot{\underline{T}}_s \cdot \underline{L}_{sr} \cdot \underline{T}_r^{-1} = \begin{bmatrix} 0 & (3/2)\omega_r L_{sr} & 0 \\ (-3/2)\omega_r L_{sr} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.31)$$

Also since $\underline{T}_r$ has zero entries:

$$\dot{\underline{T}}_r \cdot \underline{L}_{sr}^t \cdot \underline{T}_s^{-1} = \dot{T}_r \cdot \underline{L}_{rr} \cdot \underline{T}_r^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.32)$$

Next by substituting the expressions derived above in equations (3.23) through (3.32) into equation (3.22), one gets the overall state model of the induction motor in D-Q-O frame of reference fixed on the rotor as follows:

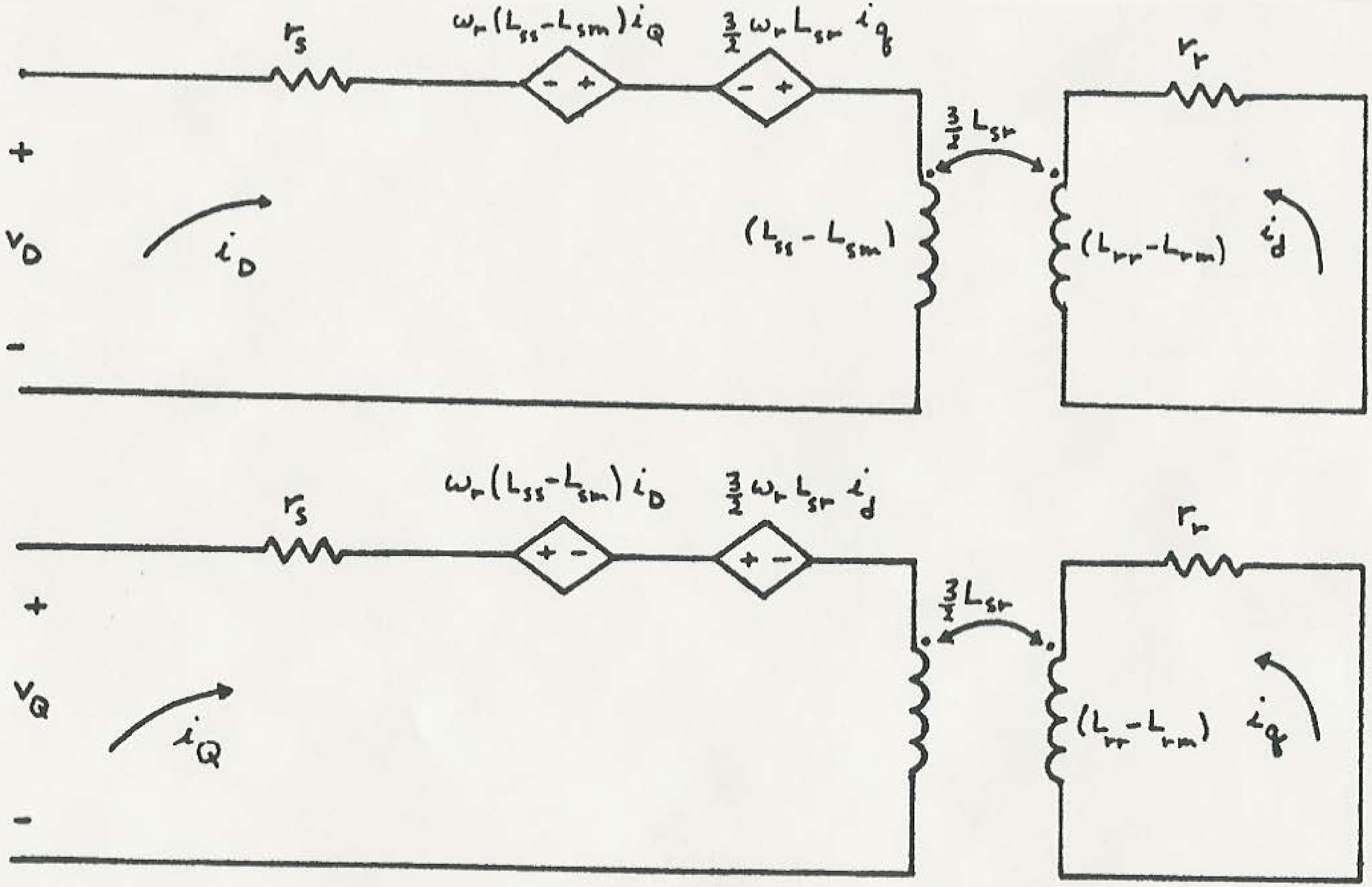
After some mathematical manipulations, the application of DQ0 transformation, with reference fixed to the rotor, would result in the following induction motor state space equation. It is in the DQ0 frame of reference, where the resulting machine inductances are constants. That is the inductances are time invariant and independent of rotor position.

$$
\begin{bmatrix} v_D \\ v_Q \\ v_0 \\ v_{d=0} \\ v_{q=0} \\ v_{o=0} \end{bmatrix}
=
\begin{bmatrix}
r_s & 0 & 0 & 0 & 0 & 0 \\
0 & r_s & 0 & 0 & 0 & 0 \\
0 & 0 & r_s & 0 & 0 & 0 \\
0 & 0 & 0 & r_r & 0 & 0 \\
0 & 0 & 0 & 0 & r_r & 0 \\
0 & 0 & 0 & 0 & 0 & r_r
\end{bmatrix}
\begin{bmatrix} i_D \\ i_Q \\ i_0 \\ i_d \\ i_q \\ i_o \end{bmatrix}
+
$$

$$
\begin{bmatrix}
(L_{ss}-L_{sm}) & 0 & 0 & (3/2)L_{sr} & 0 & 0 \\
0 & (L_{ss}-L_{sm}) & 0 & 0 & (3/2)L_{sr} & 0 \\
0 & 0 & (L_{ss}+2L_{sm}) & 0 & 0 & 0 \\
(3/2)L_{sr} & 0 & 0 & (L_{rr}-L_{rm}) & 0 & 0 \\
0 & (3/2)L_{sr} & 0 & 0 & (L_{rr}-L_{rm}) & 0 \\
0 & 0 & 0 & 0 & 0 & (L_{rr}+2L_{rm})
\end{bmatrix}
\begin{bmatrix} \dot{i}_D \\ \dot{i}_Q \\ \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_o \end{bmatrix}
$$

$$
+
\begin{bmatrix}
0 & -\omega_r(L_{ss}-L_{sm}) & 0 & 0 & (-3/2)\omega_r L_{sr} & 0 \\
\omega_r(L_{ss}-L_{sm}) & 0 & 0 & (3/2)\omega_r L_{sr} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} i_D \\ i_Q \\ i_0 \\ i_d \\ i_q \\ i_o \end{bmatrix}
\quad (3.33)
$$

Using the expressions for the different coils in equation (3.33) one can draw the following equivalent network model. The zero sequence equations equivalent network were dropped from the model since they are not coupled to the rest of the network, and since balanced 3-phase machines are being considered only.



Induction Machine DQ-dq Equivalent Circuit Model

Based on the above, state space equation of the induction machine were derived for two cases:

1. The time domain, abc, frame of reference, where the SS model is given in equation (3.14). In this case, the state variables are the currents and are in the abc frame time domain.
2. The DQ0 frame of reference, where the SS model is given in equation (3.33). In this case, the state variables are in the DQ0, time invariant domain. It is to be noted that DQ0 quantities are transformed back to the abc frame of reference using the inverse transformation matrices after completion of the computations in the DQ0 frame of reference.

The details for two case studies involving induction motor drive systems are given in the papers shown in references [1] and [2], given below.

In [1], the DQ0 frame of the reference was implemented using network circuits to represent an induction motor drive system in the DQ0 frame of reference, as was presented above. The results were transformed back to the abc frame in the time domain.

In [2], the analysis was performed directly in the time domain using the abc frame of reference and resulted in the performance characteristics of a motor drive system.

References

1. Arkadan, A.A., Johnson, V.R., and Demerdash, N.A., "A DC-AC Inverter-Induction Motor System Network Model Compatible with Commonly Known Network Analysis Software Packages," IEEE Applied Power Electronics Conference and Exposition, March 13-17, 1989, Baltimore, Maryland, Conference Proceedings, pp. 195-203.

2. Hijazi, T.M., Alhamadi, M.A., Arkadan, A.A., and Demerdash, N.A., "Modeling and Simulation of Inverter-Fed Induction Motors Using the Natural ABC Frame of Reference," Intersociety Energy Conversion Engineering Conference, August 7-12, 1989, Washington, DC, Conference Proceedings, pp. 527-534.